Teacher Created Materials
PUBLISHING

Lesson Title: Painted Cube
Foundation 4: Problem Solving and the Mathematical Practices
Mathematics Grade 7

Mathematics Common Core State Standards
Domain: Expressions and Equations 7.EE • *Cluster: Solve real-life and mathematical problems using numerical and algebraic expressions and equations* 7. EE.4—Use variables to represent quantities in a real-world or mathematical problem and construct simple equations and inequalities to solve problems by reasoning about the quantities. **Domain: Geometry 7.G** • *Cluster: Solve real-life and mathematical problems involving angle measure, area, surface area and volume* 7.G.4—Solve real world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes and right prisms.

Common Core State Mathematical Practices
MP1—Make sense of problems and persevere in solving them.
MP2—Reason abstractly and quantitatively.
MP3—Construct viable arguments and critique the reasoning of others.
MP4—Model with mathematics.
MP5—Use appropriate tools strategically.
MP6—Attend to precision.
MP7—Look for and make use of structure.

Materials

- cubes (all the same color or preferably unpainted) about 220 for each group of 2−3 students. (They should **not** be linking cubes) (2 cm cubes or 1 inch cubes are best)
- a 10 x 10 x 10 cube for the teacher
- Painted Cube Problem and Student Recording Sheet
- Earning an Allowance Student Assessment
- graph paper
- calculator (optional)
- problem-solving journals

Problem Solving Strategies

- Solve a simpler problem
- Make a model
- Make a table
- Find a pattern
- Draw a graph (optional)

Vocabulary

- volume
- exponent
- cubic units
- generalize
- surface area
- square numbers (powers of 2)
- square units
- cubic numbers (powers of 3)

(Optional vocabulary: *vertex* (pl vertices), *edge*, *face*)

Lesson Overview

1. This lesson incorporates the content standards with the mathematical practices, and at the same time develops specific problem-solving strategies needed to solve the problem. The content leads to building conceptual understanding of different types of equations and the representations when the data is graphed (linear vs. quadratic vs. cubic). Without going into great detail, students can see that when the rate of change is constant, the graph results in a line (e.g., finding the number of cubes with 2 sides painted) and how the representations change for an equation with a squared term and a cubic term indicating a much more rapid rate of change.
2. This lesson, which will take several days to complete, is divided into 3 parts. The first part introduces students to key vocabulary and reviews the mathematical concepts (i.e., volume and surface area) and skills needed to approach the new concepts. The painted problem is presented to students beginning with a 10 x 10 x 10 cube.
3. Students should be given sufficient time to attempt to solve the problem. Because a 10x10x10 cube is quite complex, students who have had experience with the strategy, *Solve a Simpler Problem*, may decide that beginning with a smaller cube makes the problem more manageable. If they have not had experience with the strategy, they may need more direction from the teacher.
4. Once students have entry into the problem, they can work in groups of 3 or 4 to continue with various sized cubes and record their findings in a table. It is important for the teacher to monitor the progress of groups and to ask questions that guide students to look for a pattern and make a generalization. Students who successfully complete the table can then graph the results. (Note that some students may not complete this part of the activity and that is okay.)

5. The third part of the lesson brings students back together to summarize and discuss their findings and solutions. This part of the lesson helps students to make connections among the mathematical ideas of the lesson and to share the patterns and strategies they found in order to solve the original problem (10 x 10 x 10) without having to actually build a 10 x 10 x 10 cube.
6. **Differentiation**: Some students may need to use the cubes to complete most of the cases. It is important that they build each cube and look for commonalities among the cubes that are painted on 1, 2, or 3 sides.
7. Most students will begin to see patterns as they build the table and after building and analyzing 3 x 3 x 3, 4 x 4 x 4 or 5 x 5 x 5 cubes, they can find the pattern and continue the table for up to 10 x 10 x 10 cases.
8. Some students will be able to generalize the pattern by seeing the relationship from one cube to the next. Others will think more algebraically, calculating the number of cubes with *x* sides painted using the dimension of the cubes.

Addressing the Common Core Mathematical Practices

- **MP1—**Make sense of problems and persevere in solving them.
  Students will be given a complex situation and use a set of simpler problems and a table to find a pattern to solve for the more complex situation.
- **MP2—**Reason abstractly and quantitatively.
  To complete the questions, students must reason and make sense of the task.
  MP3—Construct viable arguments and critique the reasoning of others.
  Students use patterns to draw conclusions and explain their thinking to others. Teacher questions and other student explanations help students to gain a clear understanding of the mathematical concepts.
- **MP4—**Model with mathematics.
  Students use concrete materials, pictures, and tables to do mathematics.
- **MP5—**Use appropriate tools strategically.
  Students will use concrete materials or drawings as needed to solve the problem. They may also use graphs and/or graphing calculators to graph the results for each case and compare the cases.
- **MP6—**Attend to precision.
  Students use appropriate and precise vocabulary as they discuss the concepts throughout the task.
- **MP7—**Look for and make use of structure.
  By starting with a simpler problem, students have the opportunity to identify and use patterns to find the solution beginning with a 1x1x1 cube (the simplest case) and building to a 10x10x10 case. Some students may be able to generalize the solution for n *cases by finding an expression to solve the problem.*

Part I: Introduction

1. Pass out copies of the Painted Cube problem to the students. Do not pass out the Painted Cube Problem Student Recording Sheet yet. Have students form groups of 3 or 4 to solve the problem. For structured interaction activities, assign each student in each group a number from 1−3 or 1−4 and assign each group a letter or a name.
2. Start by showing one cube and ask students what the volume of the cube is. Have students share their definition with a partner. Students numbered 1 and 2 should share. Students numbered 3 and 4 should share.
3. Discuss students' definitions with the whole group. Explain that the volume is the amount of cubic units it takes to fill a figure. The volume of the small cube is 1 cubic unit because the cube is 1 unit on each dimension: length, width, and height.
4. Write *volume* on the board or add it to the word wall. Have students record the definition in their Problem Solving Journals. Continue to read the problem and have students restate Mark's plan in their own words.
5. Show a painted cube (colored cube) and review with students that when the outside of the cube is painted, that is the *surface area* of the cube. Have students discuss with a partner and record their in problem-solving journals how they would calculate surface area. (Students numbered one and three and students numbered two and four should share.)
6. Discuss students' ideas about how to calculate surface area. Solidify for students that surface area is the sum of the areas of all of the faces of the figure. Further explain that surface area is measured in square units because we are actually counting the number of individual square units that are covered with paint.
7. Ask students what the surface area of one cubic unit is. Have students discuss their responses in their small groups and record their responses in their problem-solving journal. *(6 square units—each face is 1 square unit and the cube is made up of 6 faces.)* Discuss students' responses whole group.

Part II: Explore

1. Present the problem to students. Have students work in groups of 3 or 4 to solve the problem. It is likely that most students will struggle to determine the final solution for a 10 x 10 x 10 cube. If you notice that students are struggling, after about 5 minutes, bring the class back together. Ask them how they are doing and what they have noticed. Ask what is making the problem overwhelming.
2. This is a situation in which solving a simpler problem would be very helpful. Write the strategy on the board. Ask students what the simplest case of this problem would be. You may get several answers but hold out for the simplest case (1 x 1 x 1). Have students look at a single cube and determine how many cubes have exactly 6 faces painted (1), 5 faces

painted (0), 4 faces painted (0), 3 faces painted (0), 2 faces painted (0) and 1 face painted (0). It is okay if students are a bit puzzled at first!

3. Ask what the next sized cube would be (2 x 2 x 2). Have students build a 2 x 2 x 2 cube.
    - How many small cubes are in a 2 x 2 x 2? (8)
    - What is the volume of a 2 x 2 x 2 cube? (8 cubic units).
    - How many cubes have all six faces painted? (0)
    - How many cubes have five faces painted? (0)
    - How many cubes have four faces painted? (0)
    - How many cubes have three faces painted? (8)
    - How many cubes have two face painted? (0)
    - How many cubes have one face painted (0)
4. Ask students to build the next size cube (3 x 3 x 3) and continue the process of solving a series of simpler problems in their small groups. From the series of problems, direct students to look for a pattern to solve Painted Cube problem. If students have had experience making a table, let them organize their work on their own. If they do not have experience with making a table, or are struggling with organizing their work, ask them how they could keep their work organized. At this point, you may give the groups who need an organizational structure The Painted Cube Student Recording Sheet. It includes a table for organizing the work from solving the series of simpler problems.
5. Observe students work in groups to continue solving the problem for each case. Ask probing questions to facilitate students' thinking. Sample questions include:
    - Where are the cubes with 3 faces painted? 2 faces? 1 face? 0 faces?
    - What do you notice about the number of cubes with 4 sides painted?
    - When identifying the number of cubes with 0−6 faces painted, how can you be certain you have accounted for all the smaller cubes in the larger cube? *(Add them up to see if you have the total number of small cubes.)*
6. **Differentiation:**
    - Below-level students may need help putting the cubes together and completing the table. Have them solve the problem up to a 4 x 4 x 4 and ask them to describe what they notice. Other students may see the patterns and not have to build the larger cubes. It is important that the struggling students have time to build the cubes and physically point out where each type of painted cube is located. (3 faces painted will be in the corners, 2 faces painted will be along the edges between the corners, 1 face painted in the center of each face, 0 faces painted will be hidden inside.) If student notice patterns but are struggling with the computation, allow them to use calculators. Emphasize the physical patterns as well as the numerical patterns on the table.
    - On-level and above-level students can be challenged to see if they can find the solution for a 15 x 15 x 15 or a 20 x 20 x 20 cube. Challenge students to generalize a pattern for each solution (See teacher answer key).

- On-level and above-level students can graph the solutions for each set of painted sides beginning with 2 x 2 x 2 and describe the patterns on the graphs. This will show students the different shapes of the graphs based on the rate of change in the number of cubes with a specific number of faces painted verses the edge length of the larger cube.

Part III: Summarize and Discuss

1. When students have completed the table they should come together as a class to discuss their results and the patterns they see.
2. Begin by discussing the visual patterns. Review the terms *face, edge* and *vertex* and add them to the word wall. This will provide students with the vocabulary they need to discuss their observations.
   - Below are some sample questions to lead the discussion. Use the discussion strategy, Number Heads Together, to maximize student participation. Students discuss each question in small groups. They record their responses in their problem-solving journal. Then the teacher randomly selects a group using a spinner with the group names or letters on it. Then the teacher randomly selects a student in the selected group by using a spinner with the student numbers on it. The selected student in the selected group shares their response. Other students can then respond.
   - What did you notice about the small cubes with 3 faces painted?
     - *There were always 8 cubes.*
     - *They were in the vertices (corners) of the cube (vertices)*
   - Why are there always 8 cubes with three faces painted?
     - *Because there are always 8 vertices on a cube and the cubes with 3 faces painted are always positioned at the vertices of the larger cube.*
   - What did you notice about the small cubes with 2 faces painted?
     - *They were along the edges of the cube, not including the smaller cubes at the vertices.*
     - *The number of cubes on each edge was the same as the length of the cube subtract 2.*
   - Why do you have to subtract 2 from the length of the cube to determine the number of small cubes with two faces painted?
     - *There are 2 corner cubes which each had 3 faces painted?*
   - How many edges does the cube have? *(12 edges)*

- What is the pattern in the number of small cubes with 2 faces painted? *(Some students will see the number of cubes increases by 12 each time.)*

Edge Length of Large Cube	Cubes Painted on 2 Faces	First Differences	Second Differences
2	0		
3	12	12	0
4	24	12	0
5	36	12	0
6	48	12	

3. If the edge length of the large cube is known, what is an expression or series of operations that may be used to determine the number of small cubes with 2 faces painted?
   - *12*x (n–*2)*
   - *The number of edges (12) times 2 less than the length of a side.*
4. What do you notice about the small cubes with one face painted?
   - *They are always positioned in the center of each face of the large cube.*
   - *The small cubes form the shape of a square.*
   - *The length of the square is always 2 less than the length of the cube because you have already accounted for the cubes in the corners and along the edges of the cube.*
5. How many faces does the cube have? *(6 faces)*
6. What is the expression or series of operations that represents how to calculate for any size large cube the number of small cubes with one painted face?
   - $6(n-2)^2$
   - *The number of faces (6) times the square of 2 less than the length of a side.*

7. The table below shows some additional patterns. While students may not immediately recognize these patterns, you can lead a discussion that will help students see the advantage of solving a simpler problem and finding the patterns for any sized cube.

Edge Length of Large Cube	Cubes Painted on 1 Face	First Differences	Second Differences	Third Differences
2	0			
3	6	6	12	
4	24	18	12	0
5	54	30	12	0
6	96	42		

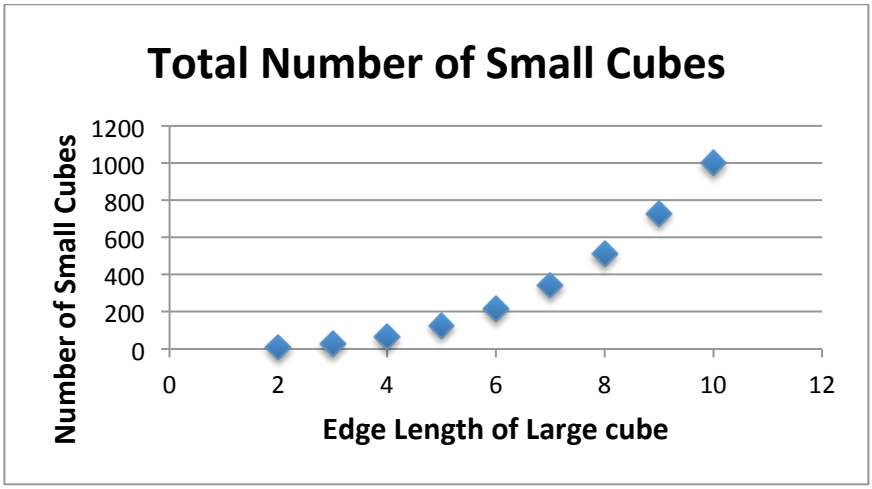
8. What did you notice about the small cubes with no faces painted?
    - *They are inside the larger cube.*
    - *They make a smaller cube inside the larger cube*
9. What are the dimensions of the smaller cube with no faces painted? *(Two less than the large cube.)*
10. Why are the dimensions of the smaller cube with no faces painted two less than the dimensions of the larger cube? *(The outer cube wraps around the cube inside, so cubes are taken away from each dimension.)*
11. Discuss the table.

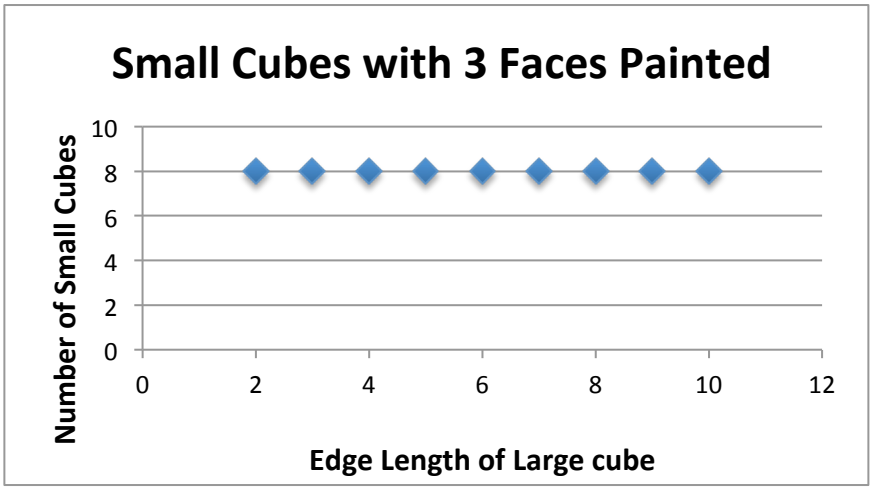
Edge Length of Large Cube	Cubes Painted on 0 Faces
2	0
3	1
4	8
5	27
6	64
7	125

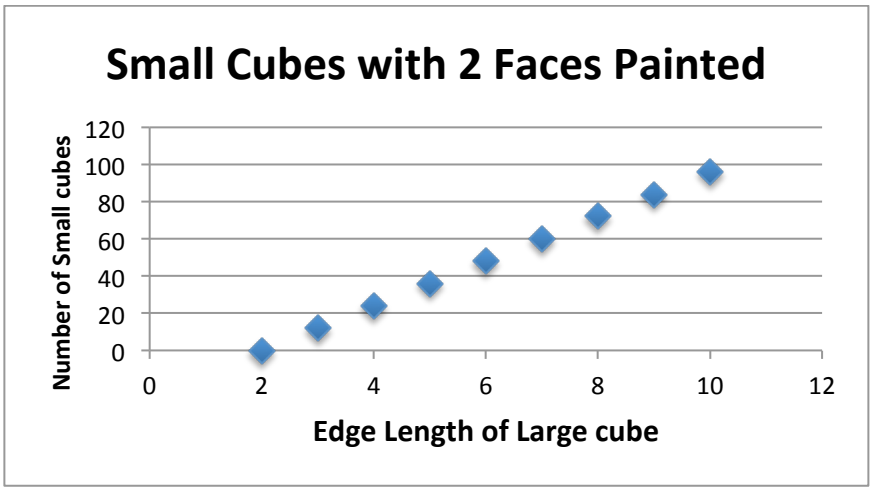
First Differences	Second Differences	Third Differences
1		
	6	
7		6
	12	
19		6
	18	
37		6
	24	
61		

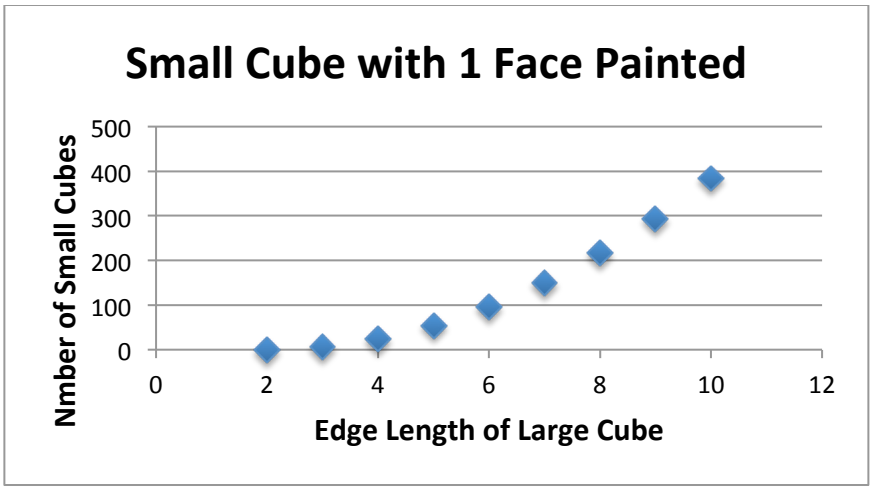
12. If students have had time to graph the results for 3, 2, 1 and 0 sides painted this would be a good time to show the graphs and compare how the graphical representation relates to the pattern descriptions, expressions, and tables. While it isn't necessary to address more abstract concepts at this time, this does give students a chance to see how the tables, expressions and graphs are different ways of representing the same mathematical ideas.
13. Display all of the graphs. Assign each group one graph with the corresponding table to analyze. Have student groups respond to the following questions related to their graph and table. They should record their answers in their problem-solving journals or on chart paper. Each group should present their answers to the class.
    - How do the graphs represent the information in the table?
    - How do the graphs represent the expression and pattern description?
14. Discuss the following questions comparing the graphs with the whole group.
    - How are the graphs similar and how they are different?
    - Which graphs shows a straight line? What does the straight line mean?
        - *For 3 sides painted, there is no change. The number of cubes is always 8. For two sides painted, there is a constant increase of 12 cubes.*
    - What do the graphs with curved lines represent?
        - *The rate of change is not the same (constant) for the next sized large cube. For 1 face painted the change in number of small cubes is exponentially greater for each larger cube. For 0 faces painted, the change in number*

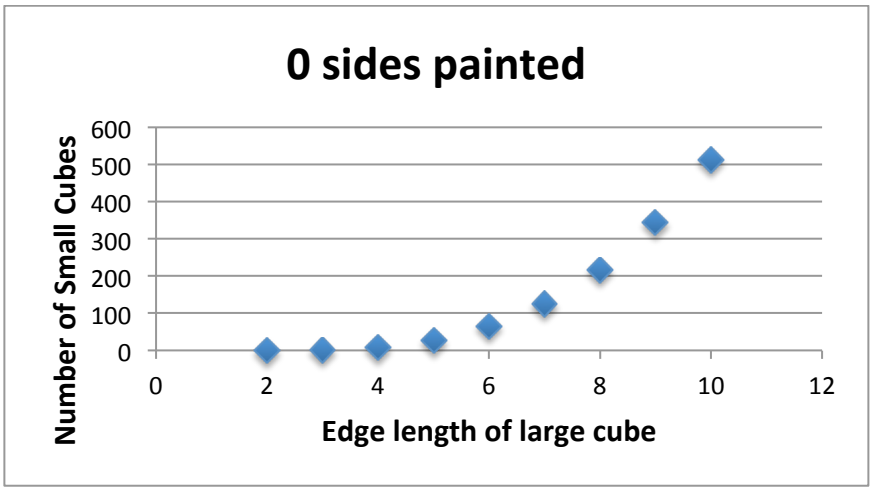
of small cubes exponentially increases at an even greater rate than 2 face painted.











Assessment

1. Have students complete the Earning an Allowance Student Assessment independently.
2. Analyze students work to see how they use the problem-solving strategies of solving a simpler problem and creating a table.
3. Analyze students' responses to see if they are able to clearly describe the pattern and represent the pattern with an expression.